Automorphism groups of von Neumann algebras and ergodic type theorems

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A great deal of work has been done for groups of $*$-automorphisms of operator algebras. Especially, relations between groups of von Neumann algebras and their invariant normal states have been studied by many authors. Above all, KOVÁCS and SZŰCS [5] showed that for a von Neumann algebra M and a group G of automorphisms of M, to have a separating family of G-invariant normal states (that is, M is G-finite), it is necessary and sufficient that there is a unique faithful normal G-invariant projection of norm 1 from M onto the fixed algebra M^G under G. In [12], E. STØRMER introduced a new equivalence (G-equivalence) on the projections in a von Neumann algebra M on a Hilbert space $\mathfrak{H}$ with a unitarily implemented group G of $*$-automorphisms which coincides with the one defined by HOPF [4] in the σ-finite abelian case and, in the general case, it includes the one due to MURRAY and VON NEUMANN [6].

Let $g \to U_g$ be a unitary representation of G on $\mathfrak{H}$ such that $U_g M U_g^* = M$ for all $g \in G$. Størmer calls two projections e and f in M G-equivalent ($e \overset{G}{\sim} f$) if for each $g \in G$ there exists an element $a_g \in M$ such that

$$\sum_{g \in G} a_g^* a_g = e \quad \text{and} \quad \sum_{g \in G} U_g^* a_g a_g^* U_g = f.$$

By using the cross product of M and G, he showed that this relation is in fact an equivalence relation, and if M is σ-finite, then M is $G\sim$finite (that is, $1 \overset{G}{\sim} e$, then $e=1$) if and only if M has a G-invariant faithful normal trace and that M is $G\sim$semi-finite (M has sufficiently many $G\sim$finite projections) if and only if M has a faithful normal G-invariant semi-finite trace.

Recently, using Ryll-Nardzewski's fixed point theorem [3, 7], F. J. YEADON gave an elegant proof of the existence of a trace in a finite von Neumann algebra [15]. It is suggestive that this method of functional analysis may be very useful

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for further studies of automorphism groups and their invariant maps of von Neumann algebras also.

The present author [8] gave a characterization of the finiteness of von Neumann algebras using weakly relatively compact subsets of their preduals and STØRMER [13] showed, roughly speaking, that a von Neumann algebra with a group G of $*$-automorphisms is G-finite if and only if for each φ in M_* (the predual of M), the orbit of φ under G is weakly relatively compact in M_*.

In this paper, we shall give some kind of a Banach space like characterization of the $G\sim$finiteness of a von Neumann algebra M with a group G of $*$-automorphisms of M, using weakly relatively compact subsets of the predual M_* which is a generalization of a theorem of HAJIAN and KAKUTANI [18, 19], more precisely to say, we shall prove

Theorem 1. *Let M be a von Neumann algebra with a group G of $*$-automorphisms of M and let $\tilde{G}$ be the group generated algebraically by G and the group of inner automorphisms of M. Then M is $G\sim$finite (see the definition below) if and only if for every weakly relatively compact subset K of the predual M_*, the set $\{\varphi\circ g\,|\,\varphi\in K, g\in\tilde{G}\}$ is also weakly relatively compact.*

The plan of this paper is as follows. Section 1 is concerned with the comparability theorem for projections relative to the $\overset{G}{\sim}$ equivalence (the abelian case was proved by STØRMER [11, Lemma 2.7]). Section 2 is devoted to the proof of Theorem 1 and contains related corollaries.

1. $\overset{G}{\sim}$ equivalence and the comparability theorem. Let M be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and G be a group of $*$-automorphisms of M defined by $a\to a^g$, $a\in M$, $g\in G$. Since we can consider G as a discrete group, there is a faithful covariant representation (U,π) of (M,G) on a Hilbert space $\mathfrak{H}$ such that $\pi(a^g)=U_g^*\pi(a)U_g$ for each $g\in G$ and $a\in M$ [20, Definition 3.1]. Since our discussions on weakly relatively compact subsets of the predual of M and others are independent of representations of M, we may always assume without loss of generality that M acts on $\mathfrak{H}$ and there is a unitary representation of G on $\mathfrak{H}$ such that $a^g=U_g^*aU_g$ for each $g\in G$ and $a\in M$.

Definition ([12]). For any pair of projections e and f in M, we write $e\overset{G}{\sim}f$ if for each $g\in G$ there exists an $a_g\in M$ such that

$$e=\sum_{g\in G}a_ga_g^*,\quad f=\sum_{g\in G}U_g^*a_g^*a_gU_g.$$

We write $e\overset{G}{\precsim}f$ or $f\overset{G}{\succsim}e$, if $e\overset{G}{\sim}e'\leqq f$ for some projection e' in M.

For each $s\in G$, let $\mathfrak{H}_s$ be a Hilbert space with the same dimension as $\mathfrak{H}$ and J_s be a linear isometry of $\mathfrak{H}$ onto $\mathfrak{H}_s$. Let $\tilde{\mathfrak{H}}$ be the direct sum of $\mathfrak{H}_s$ for $s\in G$. Any bounded linear operator R on $\tilde{\mathfrak{H}}$ is represented by a matrix $(R_{s,t})_{s\in G, t\in G}$, where $R_{s,t}=J_s^* R J_t$ (bounded linear operator on $\mathfrak{H}$). For any $a\in M$, let $\pi(a)$ be the operator on $\tilde{\mathfrak{H}}$ such that $R_{s,t}=\delta_{s,t}a$ for all s and t in G where $\delta_{s,t}$ is Kronecker's symbol. π is a $*$-isomorphism of M onto the von Neumann algebra $\{\pi(a)|a\in M\}=\tilde{M}$. For any $g\in G$, let $\tilde{U}_g$ be the unitary operator on $\tilde{\mathfrak{H}}$ defined by the matrix $(R_{s,t})$ with $R_{s,t}=0$ if $st\neq g$, $R_{gt,t}=U_g$ for all $t\in G$. Then $g\to\tilde{U}_g$ is a unitary representation of G on $\tilde{\mathfrak{H}}$ such that $\pi(U_g^* a U_g)=\tilde{U}_g^*\pi(a)\tilde{U}_g$ for all $g\in G$ and $a\in M$. Now let $M\times G$ be the von Neumann algebra generated by $\tilde{M}$ and $\{\tilde{U}_g;\ g\in G\}$. STØRMER proved the following:

Lemma 1. ([12]) *For any pair of projections in M, $e\overset{G}{\sim}f$ if and only if $\pi(e)\sim\pi(f)$ in $M\times G$ ($\sim$ is the Murray—von Neumann equivalence relation). Thus the relation $\overset{G}{\sim}$ is an equivalence relation.*

By the above lemma, $\overset{G}{\sim}$ is additive, that is, for any pair of orthogonal families of projections $\{e_\alpha\}$ and $\{f_\alpha\}$ in M with $e_\alpha\overset{G}{\sim}f_\alpha$ for each α, we have $\Sigma_\alpha e_\alpha\overset{G}{\sim}\Sigma_\alpha f_\alpha$.

Next, we shall show the comparability theorem which plays a central role in our theory and whose proof is a modification of the one given by STØRMER in the abelian case.

Proposition 1. *Let M^G be the fixed subalgebra of M under G, then for any pair e and f of projections in M, there is a projection z in $M^G\cap Z$ (Z is the center of M) such that $ez\overset{G}{\succsim}fz$ and $e(1-z)\overset{G}{\precsim}f(1-z)$.*

In order to prove this, we need some lemmas.

Lemma 2. *For any projection e in M, let $z^G(e)$ be the smallest projection in $M^G\cap Z$ majorizing e. Then $z^G(e)$ is the maximal projection f such that $f=\Sigma_\alpha f_\alpha$, $f_\alpha f_{\alpha'}=0$ $(\alpha\neq\alpha')$, and for each α, there are a projection e_α in M with $e_\alpha\leqq e$ and $g_\alpha\in\tilde{G}$ such that $U_{g_\alpha}^* e_\alpha U_{g_\alpha}=f_\alpha$.*

Proof. Let $\{f_\alpha\}$ be a maximal orthogonal family of projections in M such that for each α, there exists a projection e_α in M with $e_\alpha\leqq e$ and $g_\alpha\in\tilde{G}$ such that $U_{g_\alpha}^* e_\alpha U_{g_\alpha}=f_\alpha$. Putting $f=\Sigma f_\alpha$, we shall show $z^G(e)=f$. First we claim that $U_g f U_g^*=f$ for all $g\in\tilde{G}$. In fact, if not, there is a $g\in\tilde{G}$ such that $U_g f U_g^*(1-f)\neq 0$. Since $1-f\geqq f\vee U_g f U_g^*-f\sim U_g f U_g^*-U_g f U_g^*\wedge f\neq 0$ in usual equivalence in M, by [16, Lemma 1.7], there exist $g_0\in\tilde{G}$ and a non-zero projection $\tilde{f}$ in M such that $\tilde{f}\leqq 1-f$ and $U_{g_0}\tilde{f}U_{g_0}^*\leqq f$. $f=\Sigma f_\alpha$ implies that there is an α such that $U_{g_0}\tilde{f}U_{g_0}^* f_\alpha\neq 0$. Thus $U_{g_0}\tilde{f}U_{g_0}^*\geqq U_{g_0}\tilde{f}U_{g_0}^*-U_{g_0}\tilde{f}U_{g_0}^*\wedge(1-f_\alpha)\sim(1-f_\alpha)\vee U_{g_0}\tilde{f}U_{g_0}^*-(1-f_\alpha)\leqq f_\alpha$ in M and $(1-f_\alpha)\vee U_{g_0}\tilde{f}U_{g_0}^*-(1-f_\alpha)\neq 0$. By the same reason as above, there are a projection $\tilde{\tilde{f}}$ with $\tilde{\tilde{f}}\leqq\tilde{f}$ in M and $g'\in\tilde{G}$ such that $1-f\geqq\tilde{f}\geqq\tilde{\tilde{f}}\neq 0$ and $U_{g'}\tilde{\tilde{f}}U_{g'}^*\leqq f_\alpha$. By the definition of f_α, there is a $g_\alpha \in \tilde{G}$ such that $U^*_{g_\alpha} e_\alpha U_{g_\alpha} = f_\alpha$ for some projection e_α in M with $e_\alpha \leqq e$. Hence if we put $g_\alpha g' = \tilde{g} \in \tilde{G}$, then $U_{\tilde{g}} \tilde{f} U^*_{\tilde{g}} \leqq U_{g_\alpha} f_\alpha U^*_{g_\alpha} \leqq e_\alpha \leqq e$. Since $\tilde{f} \leqq 1-f$ and $\tilde{f} \neq 0$, $\{\tilde{f}, f_\alpha\} \supsetneqq \{f_\alpha\}$, and this contradicts the maximality of $\{f_\alpha\}$. Thus $f = U_g f U^*_g$ for all $g \in \tilde{G}$ and $f \in M^G \cap Z$. Moreover, we have $f \geqq e$. In fact, if otherwise, $1-f \geqq e \vee f - f \sim e - e \wedge f \neq 0$ (in M). Therefore, by the same argument as above, there are non-zero projections $e_1 (\leqq e - e \wedge f)$ and $f_1 (\leqq 1-f)$ in M and g in $\in \tilde{G}$ such that $e_1 = U_g f_1 U^*_g$, $\{f_1, f_\alpha\} \supsetneqq \{f_\alpha\}$, and this is a contradiction. Thus $e \leqq f$ and $z^G(e) \leqq f$. The equality $z^G(e) = f$ is shown by the following:

$$z^G(e)f = \Sigma_\alpha z^G(e) f_\alpha = \Sigma_\alpha z^G(e) U^*_{g_\alpha} e_\alpha U_{g_\alpha} = \Sigma_\alpha U^*_{g_\alpha} z^G(e) e_\alpha U_{g_\alpha} =$$
$$= \Sigma_\alpha U^*_{g_\alpha} e_\alpha U_{g_\alpha} = \Sigma_\alpha f_\alpha = f.$$

This completes the proof.

Lemma 3. *Let e and f be projections in M. If $z^G(e) z^G(f) \neq 0$, then there exist projections e_1 and f_1 in M such that $e_1 \leqq e$, $f_1 \leqq f$ and $e_1 \overset{G}{\sim} f_1$.*

Proof. By Lemma 2, there are orthogonal families $\{f_\alpha\}$ and $\{g_\beta\}$ of projections in M such that for each pair of α and β, $f_\alpha \overset{G}{\precsim} e$, $g_\beta \overset{G}{\precsim} f$, $z^G(e) = \Sigma_\alpha f_\alpha$, $z^G(f) = \Sigma_\beta g_\beta$. The assumption $z^G(e) z^G(f) \neq 0$ implies $f_\alpha g_\beta \neq 0$ for some α and β. Therefore $g_\beta \geqq (1-g_\beta) \vee f_\alpha - (1-g_\beta) \sim f_\alpha - f_\alpha \wedge (1-g_\beta) \neq 0$ in the usual equivalence in M. Since $\sim$ equivalence implies $\overset{G}{\sim}$ equivalence, we have $e \overset{G}{\succsim} f_\alpha - f_\alpha \wedge (1-g_\beta) \overset{G}{\sim} (1-g_\beta) \vee f_\alpha - (1-g_\beta) \leqq g_\beta \overset{G}{\precsim} f$, which implies that there are non-zero projections $e_1 (\leqq e)$ and $f_1 (\leqq f)$ such that $e_1 \overset{G}{\sim} f_1$. This completes the proof.

Proof of Proposition 1. Certain standard arguments show the desired property [17, Theorem 3.1.1], but for the sake of completeness we sketch them. Let $\{e_\alpha, f_\alpha\}$ be a maximal family of projections in M such that $e_\alpha e_{\alpha'} = 0$, $f_\alpha f_{\alpha'} = 0$ $(\alpha \neq \alpha')$, $e_\alpha \leqq e$, $f_\alpha \leqq f$ and $e_\alpha \overset{G}{\sim} f_\alpha$ for each α. Define $e_0 = \Sigma_\alpha e_\alpha$ and $f_0 = \Sigma_\alpha f_\alpha$, then by Lemma 1, $e_0 \overset{G}{\sim} f_0$. Putting $e_1 = e - e_0$ and $f_1 = f - f_0$, then by the maximality of $\{e_\alpha, f_\alpha\}$ we have $z^G(e_1) z^G(f_1) = 0$ by Lemma 3. Now put $g = z^G(e_1)$. Then $g \in M^G \cap Z$ and $f_1 \leqq z^G(f_1) \leqq 1 - z^G(e_1) \leqq 1-g$, which implies that $fg = f_0 g + f_1 g = f_0 g \overset{G}{\sim} e_0 g \leqq eg$ and $e(1-g) = e_0(1-g) + e_1(1-g) = e_0(1-g) \overset{G}{\sim} f_0(1-g) \leqq f(1-g)$. This completes the proof.

Using proposition 1, we have the following lemma the proof of which is an obvious modification of [10, 2.4.2].

Lemma 4. *If M is $G\sim$finite (that is, if $e \overset{G}{\sim} 1$ (e a projection), then $e=1$), then for any pair of projections f and h in M with $f \overset{G}{\sim} h$ we have $1-f \overset{G}{\sim} 1-h$.*

Proof. For $1-f$ and $1-h$, there is a projection g in $M^G \cap Z$ such that $(1-f)g \overset{G}{\precsim} (1-h)g$ and $(1-f)(1-g) \overset{G}{\succsim} (1-h)(1-g)$. Thus there is a projection f_1 in M such that $(1-f)g \overset{G}{\sim} f_1 \leqq (1-h)g$. This implies that $g = (1-f)g + fg \overset{G}{\sim} f_1 + hg \leqq g$

and by the $G\sim$finiteness of g, we have $g=f_1+hg$, that is, $f_1=g(1-h)$ and $(1-f)g \overset{G}{\sim} (1-h)g$. By the same argument, we have $(1-f)(1-g) \overset{G}{\sim} (1-h)(1-g)$ and hence $1-f \overset{G}{\sim} 1-h$. This completes the proof.

2. Proof of Theorem 1 and related corollaries. We begin with some ergodic type theorems needed in the proof of Theorem 1.

Theorem 2. *Let M be a von Neumann algebra M with a group of $*$-automorphisms and let $\tilde{G}$ be the group generated algebraically by G and the group of inner automorphisms. Then M is $G\sim$finite if and only if M is $\tilde{G}$-finite in the sense that there is a separating family of $\tilde{G}$-invariant normal functionals (G-invariant normal traces) on M, and consequently M is $G\sim$finite if and only if $M\times\tilde{G}$ is finite in the usual sense.*

The σ-finite case of the above theorem was proved by STØRMER [12, Lemma 11] by using "cross product" techniques. Using Lemma 4, the proof of Theorem 2 is a straightforward modification of the one given by YEADON [12] for finite W^*-algebras. However, due to our more complicated G-equivalence, we sketch it for the sake of completeness. First we present two lemmas.

Lemma 5. *Let $\{e_k\}$ be an increasing sequence of projections in M such that for some projection f in M, $e_k \overset{G}{\precsim} f$ for each k. Then, for $e=\sup\{e_k, k\geqq 1\}$, we have $e \overset{G}{\precsim} f$.*

Proof. Since $e_1 \overset{G}{\precsim} f$, there is a projection $f_1(\leqq f)$ in M such that $e_1 \overset{G}{\sim} f_1$. Now, by Lemma 4, $e_2 \overset{G}{\precsim} f$ implies $1-e_2 \overset{G}{\succsim} 1-f$. By additivity of G-equivalence we get $1-e_2+e_1 \overset{G}{\succsim} 1-f+f_1$, thus again by Lemma 4, we have $e_2-e_1 \overset{G}{\precsim} f-f_1$ which implies that there is a projection f_2 in M which is orthogonal to f_1 and is equivalent to e_2-e_1. Thus by mathematical induction we can choose an orthogonal sequence $\{f_i\}$ of projections in M majorized by f such that

$$\sum_{i=1}^{\infty} f_i \leqq f \quad \text{and} \quad e_{i+1}-e_i \overset{G}{\sim} f_{i+1}$$

for each i. It follows that

$$e = e_1+\sum_{i=1}^{\infty}(e_{i+1}-e_i) \overset{G}{\sim} \sum_{i=1}^{\infty} f_i \leqq f$$

and $e \overset{G}{\precsim} f$. This completes the proof.

Let M_* be the predual of M, that is the set of all ultra-weakly continuous functionals on M and let $(T_g\varphi)(a)=\varphi(U_g^* a U_g)$ $\varphi\in M_*$, $a\in M$ and $g\in\tilde{G}$. Then T_g is a linear isometry of M_* onto M_*.

Lemma 6. *For any element φ in M_*, the set $K=\{T_g\varphi \mid g\in\tilde{G}\}$ is a weakly relatively compact subset of M_*.*

Proof. If not, by [1, theorem II.2], there are an orthogonal sequence $\{e_n\}$ of projections in M, a positive real number ε and a sequence $\{g_n\}$ in $\tilde{G}$ such that

$$(*) \qquad |\varphi(U_{g_n}^* e_n U_{g_n})| \geqq \varepsilon \quad \text{for} \quad n = 1, 2, 3, \ldots .$$

Now put $f_n = U_{g_n}^* e_n U_{g_n}$. Then f_n is a projection in M such that $f_n \overset{G}{\sim} e_n$. Next we shall show that $f_n \to 0$ $(n \to \infty)$ strongly. Let $p_n = \sum_{m=n}^{\infty} e_m$ and $q_n = \bigvee_{m=n}^{\infty} f_m$, then $\{p_n\}$ and $\{q_n\}$ are monotone decreasing sequences of projections in M. For any n, put $r_k = \bigvee_{i=n}^{n+k} f_i$ for each k. Then $r_{k-1} \wedge f_{n+k} - r_{k-1} \overset{G}{\sim} f_{n+k} - r_{k-1} \wedge f_{n+k} \leqq f_{n+k} \overset{G}{\sim} e_{n+k}$ implies that $r_k = r_{k-1} \vee f_{n+k} \overset{G}{\precsim} \sum_{i=n}^{n+k} e_i \leqq p_n$ for all k, thus, by Lemma 5, $q_n = \bigvee_{k=1}^{\infty} r_k \overset{G}{\precsim} p_n$. Since M is $G\sim$finite, $1 - p_n \overset{G}{\precsim} 1 - q_n \leqq 1 - \bigwedge_{n=1}^{\infty} q_n$ for each n. Hence, again by Lemma 5, $1 = \bigvee_{n=1}^{\infty} (1 - p_n) \overset{G}{\precsim} \left(1 - \bigwedge_{n=1}^{\infty} q_n\right)$, that is $\bigwedge_{n=1}^{\infty} q_n = 0$. Thus $q_n \to 0$ $(n \to \infty)$ strongly and $q_n \geqq f_n$ for each n implies that $f_n \to 0$ (strongly) as $n \to \infty$. Hence this contradicts the inequality $(*)$, and thus K is weakly relatively compact. This completes the proof.

Proof of Theorem 2. By Krein—Šmulian's theorem [2, Theorem V.6.4], $Q(\varphi)$, the strongly closed convex hull of the weakly relatively compact set K, is weakly compact. Since the group $\{T_g : g \in \tilde{G}\}$ acting on $Q(\varphi)$ is noncontracting, that is, 0 does not belong to the closure of $\{T_g\psi_1 - T_g\psi_2 | g \in \tilde{G}\}$ whenever $\psi_1 \neq \psi_2$ and $\psi_1, \psi_2 \in Q(\varphi)$, by the Ryll-Nardzewski fixed point theorem [3, 7], there is an element $\psi \in Q(\varphi)$ such that $T_g \psi = \psi$ for all $g \in \tilde{G}$. If $\varphi \geqq 0$, then ψ can be chosen to be non-negative. For any $t \in M^{\tilde{G}} (= M^G \cap Z)$ with $t \geqq 0$, there is φ in M such that $\varphi(t) \neq 0$. For this φ, we can choose, by the above arguments, a $\psi \in Q(\varphi)$ invariant under $\tilde{G}$ such that $\psi(t) = \varphi(t) \neq 0$. Thus by [5, Proposition 1] M is $\tilde{G}$-finite, that is, M has sufficiently many $\tilde{G}$-invariant normal states. The converse assertion is clear from the definitions, so the proof is complete.

Before going into the proof of Theorem 1, we state here an ergodic type theorem.

Theorem 3. ([5]) *A von Neumann algebra M with a group G of its $*$-automorphisms is G-finite (that is, there is a separating set of G-invariant normal states on M) if and only if there is a unique faithful normal G-invariant projection of norm one from M onto the fixed algebra M^G under G.*

Proof. Let M_* be the predual of M. For every $g \in G$ and $\varphi \in M_*$, if we put $(T_g \varphi)(a) = \varphi(a^g)$ for all $a \in M$, then T_g is an order-preserving, linear isometry of M_* onto M_*. Let $Q(\varphi) = \overline{co}(T_g \varphi | g \in G)$ be the closed convex hull of $(T_g \varphi | g \in G)$ in M_*. First we shall show that $Q(\varphi)$ is weakly compact. By the Krein—Šmulian theorem

([2, Theorem V.6.4]), it suffices to prove that the bounded set $K=\{T_g\varphi|g\in G\}$ is weakly relatively compact. We claim that for any orthogonal sequence $\{p_n\}$ of non-zero projections in M, $\lim_{n\to\infty}(T_g\varphi)(p_n)=0$ uniformly for $g\in G$. In fact, if not, there is a sequence of mutually orthogonal non-zero projections $\{e_n\}$ in M, a positive number ε and a sequence $\{g_n\}$ in G such that $|\varphi(e_n^{g_n})|\geqq\varepsilon$ for all n. First of all, we shall show that $\{e_n^{g_n}\}$ is an infinite set. If not, there is a sequence of positive integers $\{n_k\}$ such that $e_{n_k}^{g_{n_k}}=e_{n_0}^{g_{n_0}}$ $(k=1, 2, 3, \dots)$ for some n_0. For any G-invariant normal state ψ,

$$\psi(e_{n_k})=\psi(e_{n_k}^{g_{n_k}})=\psi(e_{n_0}^{g_{n_0}})=\psi(e_{n_0}).$$

By the orthogonality of $\{e_{n_k}\}$, $\psi(e_{n_k})\to 0$ $(k\to\infty)$, which implies that $\psi(e_{n_0})=0$ for every G-invariant normal state ψ. Thus G-finiteness of M implies $e_{n_0}=0$, this is a contradiction and $\{e_n^{g_n}\}$ is a relatively σ-weakly compact infinite subset of M. Let a be a σ-weak cluster point of $\{e_n^{g_n}\}$, then for each positive integer i and any G-invariant normal state ϱ, there is an increasing sequence $\{n_i\}$ of positive integers such that $e_{n_i}^{g_{n_i}}\neq e_{n_j}^{g_{n_j}}$ $(i\neq j)$, and

$$|\varrho(a)-\varrho(e_{n_i}^{g_{n_i}})|<\frac{1}{i}$$

for each i. Since $\varrho(e_{g_i}^{g_{n_i}})=\varrho(e_{n_i})\to 0$ $(i\to\infty)$, we have $\varrho(a)=0$ for each G-invariant normal state ϱ, which implies, by the G-finiteness of M, that $a=0$. Since 0 is the only cluster point of $\{e_n^{g_n}\}$, we can take an element $e_{n'}^{g_{n'}}$ of $\{e_n^{g_n}\}$ such that $|\varphi(e_{n'}^{g_{n'}})|<\varepsilon$ and this is a contradiction, so $Q(\varphi)$ is weakly compact in M_*. The argument followed in the proof of Theorem 2 shows that there exists an element $\tilde{\varphi}$ in $Q(\varphi)$ such that $T_g\tilde{\varphi}=\tilde{\varphi}$ for all $g\in G$. For any $\psi\in M_*^G$ (predual of M^G) let $\varphi\in M_*$ be chosen so that $\varphi(c)=\psi(c)$ for $c\in M^G$ and $\|\varphi\|=\|\psi\|$. For this φ, as above, we can choose $\tilde{\psi}$ in $Q(\varphi)$ such that $T_g\tilde{\psi}=\tilde{\psi}$ for all $g\in G$. Next we shall show that the $\tilde{\psi}$ is uniquely determined by ψ. In fact, let ϱ be in M_* such that $\varrho(x)=\psi(x)$ for all $x\in M^G$ and $\varrho(x^g)=\varrho(x)$ $(g\in G, x\in M)$. Let $\sigma=\varrho-\tilde{\psi}$, and let $\sigma=v|\sigma|$ be the polar decomposition of σ ([10, 1.14.4]). Then $(T_g\sigma)(x)=\sigma(x^g)=|\sigma|(x^g v)=|\sigma|((xv^{g^{-1}})^g)=(T_g|\sigma|)(xv^{g^{-1}})=$ $=v^{g^{-1}}T_g|\sigma|(x)$ for all $x\in M$. On the other hand, since $\|T_g|\sigma|\|=\||\sigma|\|=\|\sigma\|=\|T_g\sigma\|$ and $T_g|\sigma|\geqq 0$, we have $|T_g\sigma(x)|^2\leqq(T_g|\sigma|)(xx^*)\|T_g|\sigma|\|$. By the unicity of polar decomposition, $T_g\sigma=\sigma$ $(g\in G)$ implies, that $T_g|\sigma|=|\sigma|$, $v^{g^{-1}}=v$ for all $g\in G$ and $\|\sigma\|=\||\sigma|\|=|\sigma|(1)=\sigma(v^*)=\varrho(v^*)-\psi(v^*)=0$ $(v\in M^G)$. Thus $\varrho=\psi$. If we apply the above argument with ψ, then we have $\|\psi\|=\|\tilde{\psi}\|$ and if $\psi\geqq 0$, then $\|\psi\|=\psi(1)=\tilde{\psi}(1)=$ $=\|\tilde{\psi}\|$ which implies $\tilde{\psi}\geqq 0$. Put $\Phi_*\psi=\psi$. Then Φ_* is a positive, linear isometric mapping of M_*^G to M_* such that

(a) $$T_g(\Phi_*\psi)=\Phi_*\psi \quad \text{for all} \quad g\in G,$$

(b) $$(\Phi_*\psi)(x)=\psi(x) \quad \text{for all} \quad x\in M^G.$$

Now denote the transposed mapping of Φ_* by Φ_G, then we can easily show that Φ_G is a faithful G-invariant normal projection of norm one from M onto M^G. It is easy to show the converse assertion. This completes the proof.

Remark. The last half part of the proof of the above theorem is a slight modification of [15, Theorem].

Now we are in the position to prove Theorem 1. The following lemma is essential for our discussions.

Lemma 8. *Keeping the notations and assumptions in Theorem 1 in mind, let $\{\varphi_i\}$ be a sequence in M_* such that φ_i converges weakly to φ_0 in M_*. If M is $G\sim$finite, then for arbitrary sequence $\{a_n\}$ in the unit sphere S of M such that $a_n\to 0$ $(n\to\infty)$ strongly, we have $\lim_{n\to\infty}\varphi_i(a_n^g)=0$ uniformly for $i=1, 2, 3, \dots$ and for $g\in\tilde{G}$.*

Proof. Put $\varphi=\sum_{i=1}^{\infty}\frac{|\varphi_i|}{2^i\|\varphi_i\|}$ (where $|\varphi_i|$ is the absolute value of φ_i [10, 1.14.4]). Then $\varphi\in M_*$. Let $z^G(e_\varphi)$ be the least projection in $M^G\cap Z$ majorizing e_φ (the support projection of φ), then $Mz^G(e_\varphi)$ is σ-finite. In fact, since $z^G(e_\varphi)\in M^G\cap Z$, we may suppose $\tilde{G}$ is a group of $*$-automorphisms of $Mz^G(e_\varphi)$ and the fixed subalgebra of $Mz^G(e_\varphi)$ under $\tilde{G}$ is $(M^G\cap Z)z^G(e_\varphi)$. By Theorem 3, the $\tilde{G}$-finiteness of $Mz^G(e_\varphi)$ implies that there is a faithful normal $\tilde{G}$-invariant projection of norm one from $Mz^G(e_\varphi)$ onto $(M^G\cap Z)z^G(e_\varphi)$. Thus to prove the above assertion, we only have to show that $(M^G\cap Z)z^G(e_\varphi)$ is σ-finite. To do this, let e' be the support projection of the restriction of φ on $M^G\cap Z$, then $e'\geqq z^G(e_\varphi)$, which implies the σ-finiteness of $(M^G\cap Z)z^G(e_\varphi)$. An easy calculation shows that $\varphi_i(a)=\varphi_i(az^G(e_\varphi))$ for all i and $a\in M$, and hence to prove the lemma, we may assume that M is σ-finite. Thus by the $\tilde{G}$-finiteness of M (Theorem 2), there is a faithful $\tilde{G}$-invariant normal state (G-invariant normal trace) τ on M. We define a metric $d(x, y)$ on S as $d(x, y)=$ $=\tau((x-y)^*(x-y))^{1/2}$. Then (S, d) is a complete metric space which is equivalent to (S, strong topology). For each positive integer i, let

$$H_i=\{a\,|\,a\in S,\ |\varphi_j(a)-\varphi_0(a)|\leqq\varepsilon\quad\text{for all}\quad j\geqq i\}.$$

Then H_i is a closed subset of S for each i and $S=\bigcup_{i=1}^{\infty}H_i$. Baire's category theorem implies that there exists $a_0\in S$, a positive number β and a positive integer j_0 such that $\{a\,|\,\|a\|\leqq 1,\ \tau((a-a_0)^*(a-a_0))^{1/2}\leqq\beta\}\subset H_{j_0}$. Since $a_n\to 0$ $(n\to\infty)$ strongly and M is finite, the self-adjoint and skew-adjoint parts of $\{a_n\}$ both converge strongly to 0, so that we can suppose that each a_n is self-adjoint. Thus by spectral theory, there is a sequence $\{e_n\}$ of projections in M such that $e_n\to 1$ $(n\to\infty)$ strongly and $\|a_ne_n\|\leqq\varepsilon/6$ for each n. Thus, by the fact that $\|(a_ne_n)^g\|=\|a_ne_n\|$ for each $g\in\tilde{G}$, we get that

$$|(\varphi_j-\varphi_0)(a_n^g)| \leqq |(\varphi_j-\varphi_0)(e_n^g a_n^g e_n^g)|+\left|(\varphi_j-\varphi_0)\big(e_n^g a_n^g(1-e_n^g)\big)\right|+$$
$$+\left|(\varphi_j-\varphi_0)\big((1-e_n^g)a_n^g e_n^g\big)\right|+\left|(\varphi_j-\varphi_0)\big((1-e_n^g)a_n^g(1-e_n^g)\big)\right| \leqq$$
$$\leqq \varepsilon(\sup_j \|\varphi_j\|)+\left|(\varphi_j-\varphi_0)\big((1-e_n^g)a_n^g(1-e_n^g)\big)\right|.$$

Now put $b_n(g)=e_n^g a_0 e_n^g+(1-e_n^g)a_n^g(1-e_n^g)(\in S)$. Then by the $\tilde{G}$-invariant of τ, it follows that

$$\tau\big((b_n(g)-a_0)^*(b_n(g)-a_0)\big)^{1/2} \leqq \tau\big((1-e_n^g)a_n^g(1-e_n^g)a_n^g(1-e_n^g)\big)^{1/2}+$$
$$+\tau\big((1-e_n^g)a_0^*(1-e_n^g)a_0(1-e_n^g)\big)^{1/2}+\tau\big((1-e_n^g)a_0^* e_n^g a_0(1-e_n^g)\big)^{1/2}+$$
$$+\tau\big(e_n^g a_0^*(1-e_n^g)a_0 e_n^g\big)^{1/2} \leqq 3\tau(1-e_n^g)^{1/2}+\tau\big(e_n^g a_0^*(1-e_n^g)a_0 e_n^g\big)^{1/2} =$$
$$= 3\tau(1-e_n^g)^{1/2}+\tau\big((1-e_n^g)a_0 e_n^g a_0^*(1-e_n^g)\big)^{1/2} \leqq 4\tau(1-e_n^g)^{1/2} = 4\tau(1-e_n)^{1/2}.$$

Since $e_n\to 1$ $(n\to\infty)$ strongly, there is a natural number $n_0(\beta)$ (independent of $g\in\tilde{G}$) such that $\tau\big((b_n(g)-a_0)^*(b_n(g)-a_0)\big)^{1/2}\leqq 4\tau(1-e_n)^{1/2}<\beta$ for all $n\geqq n(\beta)$. Thus we have that

$$\left|(\varphi_j-\varphi_0)(e_n^g a e_n^g)+(\varphi_j-\varphi_0)\big((1-e_n^g)a_n^g(1-e_n^g)\big)\right| \leqq \varepsilon$$

for all $j\geqq j_0$, for all $g\in\tilde{G}$ and for all $n\geqq n_0(\beta)$. By the same argument as above it follows that

$$\tau\big((e_n^g a_0 e_n^g-a_0)^*(e_n^g a_0 e_n^g-a_0)\big)^{1/2} \leqq 3\tau(1-e_n)^{1/2},$$

so that there exists a natural number $n_1(\beta)$ (independent of $g\in\tilde{G}$) such that

$$\tau\big((e_n^g a_0 e_n^g-a_0)^*(e_n^g a_0 e_n^g-a_0)\big)^{1/2} \leqq 3\tau(1-e_n)^{1/2} < \beta$$

for all $n\geqq n_1(\beta)$. Hence, observing that $e_n^g a_0 e_n^g\in S$, we get that

$$|(\varphi_j-\varphi_0)(e_n^g a_0 e_n^g)| \leqq \varepsilon$$

for all $j\geqq j_0$, all $g\in\tilde{G}$ and for all $n\geqq n_1(\beta)$. Thus combining the above estimates, we get

$$\left|(\varphi_j-\varphi_0)\big((1-e_n^g)a_n^g(1-e_n^g)\big)\right| \leqq 2\varepsilon$$

for all $j\geqq j_0$, $g\in\tilde{G}$ and $n\geqq \max\big(n_0(\beta), n_1(\beta)\big)$, which implies that

$$|(\varphi_j-\varphi_0)(a_n^g)| \leqq \varepsilon(\sup_i \|\mathring{\varphi}_i\|)+2\varepsilon$$

for all $j\geqq j_0$, $g\in\tilde{G}$ and $n\geqq\max\big(n_0(\beta), n_1(\beta)\big)$. Now let $K=\{T_g(\varphi_j-\varphi_0), T_g\varphi_0 \mid j=1, 2, 3, \dots, j_0-1; g\in\tilde{G}\}$. Then by the proof of Theorem 3, K is a weakly relatively compact bounded subset of the predual M_* of M. Then by [1, Theorem II.3], there is a normal functional ψ on M such that for every positive number δ, there exists a positive number γ such that the inequality $\psi(a^*a+aa^*)<\gamma$ $(a\in S)$ implies that

$$|T_g(\varphi_j-\varphi_0)(a)| < \delta$$

and

$$|(T_g\varphi_0)(a)| < \delta$$

for all $g\in\tilde{G}$ and $j=1, 2, \ldots, j_0-1$. Since $a_n\to 0$ $(n\to\infty)$ strongly, take $\delta=\varepsilon$. Then for γ determined above, there is a natural number $n_2(\varepsilon)$ such that $\psi(a_n^* a_n+a_n a_n^*)<\gamma$ for all $n\geqq n_2(\varepsilon)$. Thus it follows that

$$|(\varphi_j-\varphi_0)(a_n^g)| < \varepsilon, \quad |\varphi_0(a_n^g)| < \varepsilon$$

for all $g\in\tilde{G}$, $j=1, 2, \ldots, j_0-1$ and for all $n\geqq n_2(\varepsilon)$. Hence combining the above estimates, we have

$$|\varphi_j(a_n^g)| \leqq \varepsilon(\sup_i \|\varphi_i\|)+3\varepsilon$$

for all $g\in\tilde{G}$ and $j=1, 2, 3, \ldots$ and for all $n\geqq\max\big(n_0(\beta), n_1(\beta), n_2(\beta)\big)$, which implies that $\varphi_j(a_n^g)\to 0$ $(n\to\infty)$ uniformly for $g\in\tilde{G}$ and j. This completes the proof.

Proof of Theorem 1. Suppose M is $G\sim$finite. If K is any weakly relatively compact subset of M_*, then $\{T_g\varphi\,|\,\varphi\in K, g\in\tilde{G}\}$ is also weakly relatively compact. In order to prove this, we only have to prove that for every orthogonal sequence $\{e_n\}$ of projections, $\lim_{n\to\infty}\varphi(e_n^g)=0$ uniformly for $g\in\tilde{G}$ and $\varphi\in K$. If not, there is a positive number ε such that for each positive integer k, there are a natural number n_k $(n_k\uparrow\infty)$, $g_k\in\tilde{G}$ and $\varphi_k\in K$ such that

$$(*) \qquad |\varphi_k(e_{n_k}^{g_{n_k}})| \geqq \varepsilon.$$

By the Eberlein—Šmulian theorem ([2], Theorem V.6.1) there is a subsequence $\{\varphi_{k_p}\}$ of $\{\varphi_k\}$ $(k_p\uparrow\infty)$ such that $\varphi_{k_p}\to\varphi_0$ weakly $(p\to\infty)$ for some $\varphi_0\in M_*$. Now $e_{n_{k_p}}\to 0$ $(p\to\infty)$ strongly implies, by Lemma 8, that

$$\varphi_{k_{p'}}(e_{n_{k_p}}^g) \to 0 \quad (p\to\infty)$$

uniformly for $p=1, 2, 3, \ldots$ and $g\in\tilde{G}$ and that

$$|\varphi_{k_p}(e_{n_{k_p}}^{g_{k_p}})| \to 0 \quad (p\to\infty)$$

contradicting the inequality $(*)$. Hence $\{T_g\varphi\,|\,\varphi\in K, g\in\tilde{G}\}$ is weakly relatively compact. The converse is clear from Theorems 2 and 3. This completes the proof of Theorem 1.

The following corollary concerns another characterization of finiteness of a von Neumann algebra ([8], Theorem 1 and the Remark following Theorem 3, more precisely, a von Neumann algebra M is finite if and only if for every weakly relatively compact subset K of the predual M_* of M, the set $\{a\varphi\,|\,a\in S, \varphi\in K\}$ is also weakly relatively compact (where $(a\varphi)(x)=\varphi(xa)$ for all $x\in M$). In the proof of this assertion, the equivalence of the Mackey topology and the strong topology on S for finite algebras [9] plays an essential rôle.

Corollary 1. *For a von Neumann algebra M to be finite, it is necessary and sufficient that for every weakly relatively compact subset K of M_* (predual of M) the set*

$$\{u\varphi u^* \mid \varphi \in K,\ u \text{ is a unitary operator of } M\}$$

(where $(u\varphi u^*)(x) = \varphi(u^* x u)$ $x \in M$) is also weakly relatively compact.

Proof. Let $\mathfrak{U}$ be the group of inner automorphisms of M. Then if we apply Theorem 1 to M with $\mathfrak{U}$, we can easily show the above statement.

Corollary 2. *Let M be a finite von Neumann algebra with a group G of $*$-automorphisms of M. Then M is $G\sim$finite if and only if M is G-finite and if and only if for every weakly relatively compact subset K of M_*, $\{T_g\varphi \mid \varphi \in K, g \in G\}$ is also weakly relatively compact.*

Proof. If M is G-finite, then there is a separating set $\{\varphi_\alpha\}$ of G-invariant normal states on M. Since $\varphi_\alpha | Z$ (the restriction of φ_α to Z) is G-invariant for each α, Z is also G-finite. Let Φ be the center valued trace on M, then by unicity it follows that $\Phi(a^g) = \Phi(a)^g$ for all $g \in G$ and $a \in M$. Hence $\{(\varphi_\alpha | Z) \circ \Phi\}$ is a separating set of G-invariant normal traces on M and $G\sim$finiteness of M follows. The rest of the above assertions follows from Theorems 1, 2 and 3. The proof is now completed.

In particular, in the abelian case, we have a necessary and sufficient condition for the existence of invariant measures in ergodic theory, closely related to the theorem of Hajian and Kakutani [18, 19].

Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space and let G be a group operating on the left on X by $\zeta \to s\zeta$, $\zeta \in X$ and suppose μ is quasi-invariant, that is, $\mu(sE) = 0$ if and only if $\mu(E) = 0$ for $E \in \mathfrak{M}$. Let $r_s(\cdot)$ be the Radon—Nikodym derivative such that $d\mu(s\zeta) = r_s(\zeta)\, d\mu(\zeta)$.

Corollary 3. *In the above notations, for $(X, \mathfrak{M}, \mu)$ to have a faithful G-invariant measure, it is necessary and sufficient that for every weakly relatively compact subset K of $L^1(X, \mathfrak{M}, \mu)$ (the set of all μ-integrable complex-valued functions on X)*

$$\{r_{s^{-1}}(\cdot) f(s^{-1}\cdot) \mid f \in K, g \in G\}$$

is also weakly relatively compact in $L^1(X, \mathfrak{M}, \mu)$.

Remark. Hopf [4] proved that $(X, \mathfrak{M}, \mu)$ has a faithful G-invariant measure if and only if $(X, \mathfrak{M}, \mu)$ is Hopf-finite in the sense that if for $E \in \mathfrak{M}$ there are countable families $\{E_i\}$ in $\mathfrak{M}$ and $\{g_i\}$ in G such that

$$X = \bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad E = \bigcup_{i=1}^{\infty} g_i E_i, \quad \text{then} \quad \mu(X - E) = 0.$$

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